How Much Uncertainty can Feedback Mechanism Deal with ?

Lei GUO

Institute of Systems Science, AMSS,
Chinese Academy of Sciences
(Lguo@iss.ac.cn)

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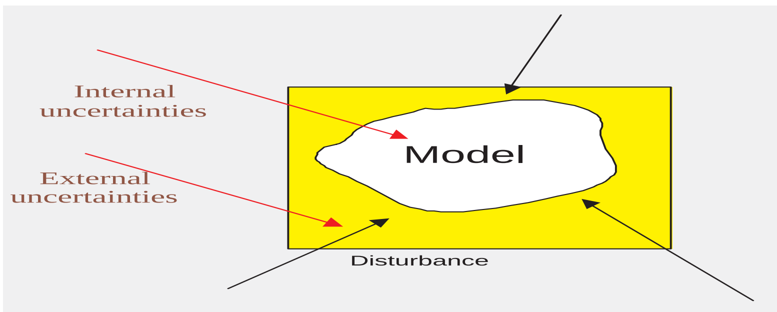
Outline

I. Background

Feedback

- Needed in almost all purposive behaviors.
- A basic systems principle and a core concept in automatic control.
- **Why feedback ?** Uncertainties always exist in modeling, controlling and running of practical systems.



Feedback Theory

The feedback control of uncertain dynamical systems has been a central theme in control theory, and tremendous progress has been made in, e.g., robust, adaptive, nonlinear and stochastic control, etc.

Typical techniques developed include:

PID control,
Bode's design method,
Internal model principle,
Gain scheduling, high-gain,
Nonlinear damping,
Stochastic dynamic programing,
Geometric control theory,
LMI, MPC,
ISS, passivity theory
Sliding mode control,
Absolute stability,
....

H_∞ **control,**
Self-tuning regulators,
MRAC,
Small-gain theorems,
Kharitonov theorem,
Kalman filtering,
μ**-design,**
Back-stepping method,
Neural network and Fuzzy control,
Dual principle,
Active-disturbance rejection control,

Expectations from Theory

- **Any theory needs assumptions, and any mathematical models are simplifications/approximations of real systems.**

- **"Make everything as simple as possible, but not simpler"(A.Einstein)**

- **While it may be necessary to have a theory established for simplified models, one may naturally expect that a theory can give the boundaries of its applicability to real systems, which are mostly nonlinear with uncertainties.**

$$G(\cdot) = g(\theta, \cdot) + f(\cdot), \qquad f(\cdot) = [G(\cdot) - g(\theta, \cdot)]$$

Some Challenges(I)

- **Implementation of Control**: Feedback cannot be implemented instantaneously in general, due to sensing, sampling, communication, computation, actuation and so on. This turns out to be quite challenging in theory.

- **Identification for Control**: Is identification really necessary for control? How much uncertainty should at least be reduced by modeling and identification, in order that the remaining uncertainty can be effectively dealt with by feedback?

Some Challenges (II)

- **Model-Based Control: Can we say anything about the maximal nonlinear uncertainty that can be dealt with by controllers designed based on a given model class ?**

$$G(\cdot) = g(\theta, \cdot) + f(\cdot), \qquad f(\cdot) = [G(\cdot) - g(\theta, \cdot)]$$

- **Data-Driven Control: Do we know any limitations of controllers that are designed on the basis of the online observed or measured data ?**

Some Challenges(III)

- **Adaptive Nonlinear Control: Can we design a stabilizing adaptive controller for discrete-time uncertain systems with nonlinearities having a growth rate faster than linear? If cannot, is it due to our personal incapability or due to fundamental limitations of the feedback mechanism ?**

- **Hybrid Nonlinear Control: Can we have some concrete results on sampled-data control of nonlinear uncertain systems, when the sampling rate is prescribed ?**

Related Problems

- **How much uncertainty can the feedback mechanism deal with ?**
- **What are the limitations of the feedback mechanism?**
- **How to construct the most powerful feedback law?**
- **How does uncertainty affect the feedback performance?**

Related Directions

- **Much progress has been made in**

  Robust control

  Adaptive control

  . . . . . .

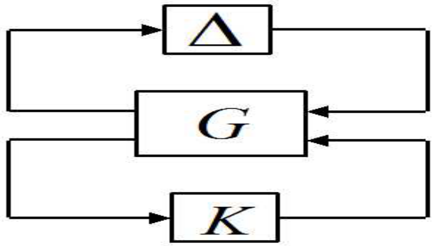
- **Few results address**

  the maximum capability and limitations

  of the feedback mechanism

Robust Control

- Model = Nominal + "ball"



- Control law design

  Based on the nominal model and from a given class

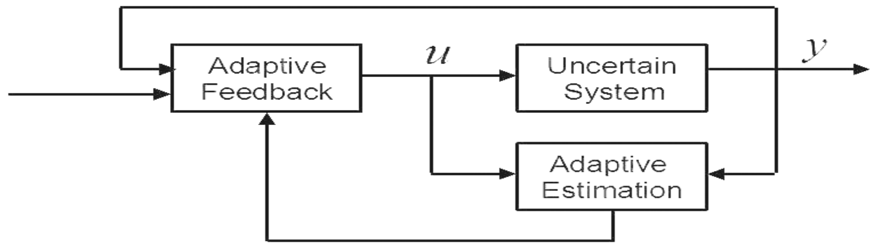
  Measurements are not used to reduce the "Ball"

Related but essentially different

in both problems and results

Adaptive Control

- **A standard situation**



- **Typical Design Methods**

  Certainty equivalence principle

  Lyapunov function based

Does not address the
maximum capability and limitations

II. Problem Formulation

Feedback and Information

Information = prior + posterior

$$= I_0 + I_t$$

$I_0 =$ prior knowledge about the plant

$I_t =$ posterior knowledge about the plant

$= \{y_0, y_1, \cdots, y_t\}$ (Observations/Data)

Definition of Feedback

- **Feedback signal u_t :** there is a measurable mapping

$$f_t : \mathbb{R}^{t+1} \to \mathbb{R}^1$$

such that

$$u_t = f_t(y_0, y_1, \cdots, y_t)$$

- **Feedback law u :**

$$u = \{u_t, \quad t \geq 0\}$$

- **Feedback mechanism U:**

$$U = \{u \,|\, u \text{ is any feedback law}\}$$

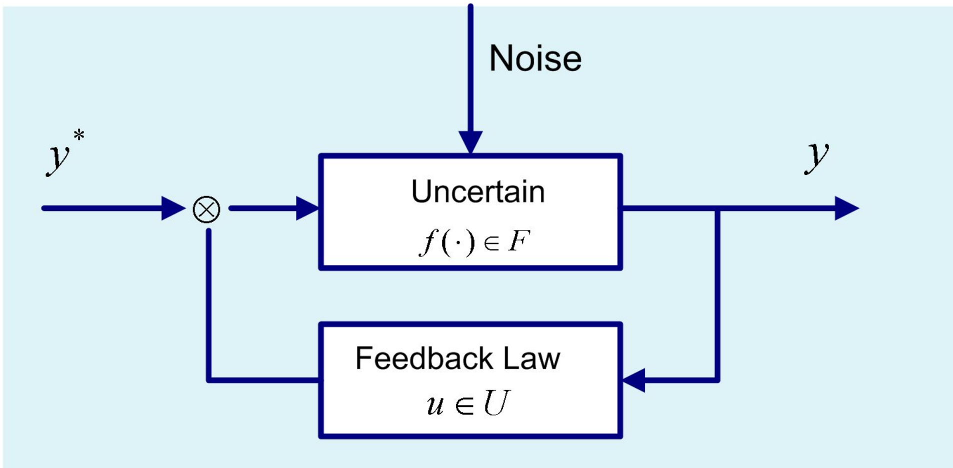
Description of Uncertainty

- **Uncertainty is mathematically described by a set, either parametric or functional.**

- **The control of uncertain systems is by definition the control of all possible systems related to this set.**

(Plenty but not Unnecessary)

Problem Formulation



$$\sup_{F} \left\{ \mathbf{size}(F) : \inf_{u \in U} \sup_{f \in F} \sup_{t \geq 0} |y_t(f, u)| < \infty, \quad \forall\, y_0 \in \mathbb{R} \right\}$$

Basic Questions

- **How to properly define the uncertainty set $\mathcal{F}$?**
- **How to characterize the limits of feedback?**
- **How to construct the most powerful feedback?**
- **How to analyze such dynamical systems?**

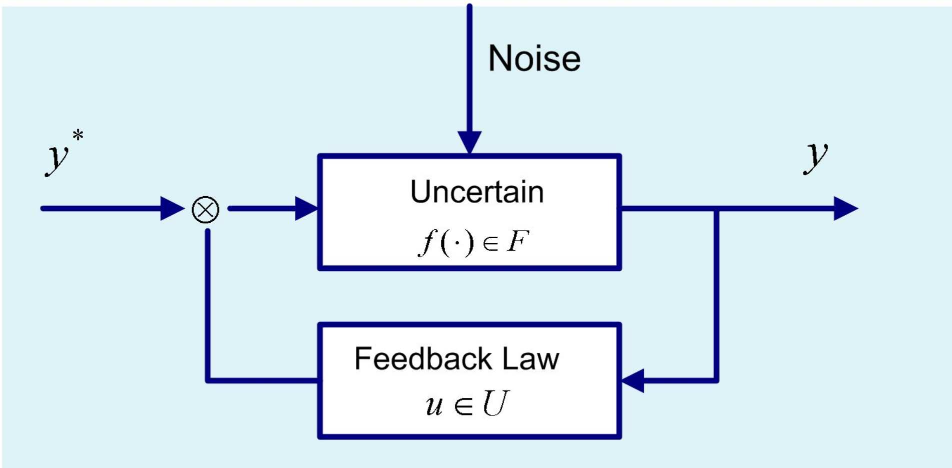
(cont.)

Basic Questions

- **How to properly define the uncertainty set $\mathcal{F}$?**
- **How to characterize the limits of feedback?**
- **How to construct the most powerful feedback?**
- **How to analyze such dynamical systems?**

Even if a nonlinear feedback law works well both numerically and practically, the last question may still not be easy in general, just like the case of self-tuning regulators.

On Impossibility Theorems



A Basic Fact: Let $\mathcal{F}_0$ and $\mathcal{F}$ be two classes of uncertain systems satisfying

$$\mathcal{F}_0 \subset \mathcal{F}.$$

If the system class $\mathcal{F}_0$ cannot be stabilized by the feedback mechanism, then neither for the larger class $\mathcal{F}$.

Some Known Impossibility Theorems

- **Bode Integral Theorem on Sensitivity Functions (1940);**
- **Shannon Channel Coding Theorem (1948) ;**
- **Cramér-Rao Bound in Statistics (1945);**
- **Heisenberg Uncertainty Principle in Physics (1927);**
- **Gödel Incompleteness Theorem in Mathematics (1931);**
- **Arrow Impossibility Theorem in Economics (1951).**
- **... ... ...**

Feedback Capability

- **Fundamental Limitations:** Can prevent us from wasting time and energy on searching for an expected feedback controller that does not exist, and alert us of the danger of being unable to control uncertain systems when the size of the uncertainty reaches the limit established.

- **Maximum Capability:** Can encourage us in improving the controller design to reach or approximate the maximum capability, and may also guide and help us in relaxing the task and efforts in mathematical modeling and system identification.

A Basic Model Class

$$y_{t+1} = f(y_t) + u_t + w_{t+1}, \quad y_0 \in \mathbb{R}^1$$

where

- $\{y_t\}$ and $\{u_t\}$ are output and input sequences;
- $\{w_t\}$ is any bounded disturbance (or white noise);
- $f(\cdot)$ belongs to a class $\mathcal{F}$ with uncertainty.

Question: How much uncertainty in $f(\cdot) \in \mathcal{F}$ can be dealt with by feedback?

Uncertainty and Nonlinearity

- **Uncertainty:**

  Parametric: **the uncertainty can be described by a finite number of unknown parameters, either nonlinearly parametric $f(\theta, x)$ or linearly parametric $\theta^\tau f(x)$;**

  Nonparametric: **Functional uncertainty.**

- **Nonlinearity:**

  Growth Rate: $\boldsymbol{f(x) = \Theta(|x|^b)}, \quad \boldsymbol{b \geq 0}, \quad x \to \infty,$

which means that there exist positive constants c_1 and c_2 such that $c_1 \leq |f(x)|/|x|^b \leq c_2$ for all sufficiently large x.

Sensitivity Function(SF)

$$\frac{\partial f(\theta, x)}{\partial \theta}$$

- **Remark:** **If the change of θ has no "significant" influence on $f(\theta, x)$, then there is no need to care about it. In fact, the growth rate of SF is more relevant than the range of the parameter change, and the following quantity will play a key role:**

$$C_f \triangleq \limsup_{x \to \infty} \log \left| \frac{\partial f(\theta, x)}{\partial \theta} \right| / \log x$$

III. Some Fundamental Theorems

Case I. Polynomial Criteria

The Critical Value $b = 4$

$$y_{t+1} = f(\theta, y_t) + u_t + w_{t+1}.$$

Assume that the sensitivity function satisfies

$$\frac{\partial f(\theta, x)}{\partial \theta} = \Theta(|x|^b), \quad x \to \infty, \quad b \geq 0,$$

where the unknown parameter $\theta \in \mathbb{R}^1$ lies in a compact set and $\{w_t\}$ is bounded disturbance.

> **Theorem ($b = 4$ is critical):**
> **The above class of systems is globally stabilizable by feedback if and only if $\boldsymbol{b < 4}$.**

(Li & Guo, IEEE-TAC, 2011)

A Corollary

Linear parameter case:

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1}.$$

Assume that for some $b \geq 0$,

$$f(x) = \Theta(|x|^b), \quad x \to \infty,$$

and that $\theta \in \mathbb{R}^1$ is unknown and $\{w_t\}$ is either white noise or any bounded noise.

Theorem ($b = 4$ is critical):
The above class of systems is globally stabilizable by feedback if and only if $b < 4$.

(Guo, IEEE-TAC, 1997; Li & Xie, SCL, 2006)

Why $b = 4$?

- **The noise effect is essential. If there were no noise, we would have $\theta = (y_2 - u_1)/f(y_1)$, and consequently, the systems could be stabilized trivially, regardless of the value of $b > 0$.**

- **In the noise case where $\{w_t\}$ is assumed to be white , $f(y_t) = y_t^b$, the "best" estimate (LS) $\hat{\theta}_t$ will give a natural feedback $u_t = -\hat{\theta}_t y_t^b$, and so**

$$y_{t+1} = \tilde{\theta}_t y_t^b + w_{t+1}.$$

Notice that the estimation error $\tilde{\theta}_t \triangleq \theta - \hat{\theta}_t$ will decrease at a rate roughly $1/y_{t-1}^b$, as long as $\{y_t^b\}$ increases fast enough. This will imply that the critical case for stability is $b = 4$.

Why $b = 4$? (Cont'd)

- **The rigorous analysis for the general case hinges on the following fact: for any positive sequence $\{S_t\}$ with**

$$S_{t+1} \leq C + \sum_{i=1}^{t} \delta_i \left(\frac{S_i}{S_{i-1}} \right)^b, \quad \sum_{i=1}^{\infty} \delta_i < \infty,$$

$\{S_t\}$ is bounded provided that $b < 4$.

- **The above result is connected to the following fact:**

$$z^2 - b_1 z + b_1 > 0, \quad \forall z \in (1, b_1)$$

if and only if $b_1 < 4$.

The Critical Value $b = 3$

Additional uncertainty in the input channel:

$$y_{t+1} = \theta_1 f(y_t) + \theta_2 u_t + w_{t+1}, \quad y_0 \in \mathbb{R}^1,$$

- (θ_1, θ_2) **belongs to a compact set with** $\theta_2 \neq 0$; $\{w_t\}$ **is any bounded noise.**
- $f(x) = \Theta(|x|^b)$ **as** $|x| \to \infty$ **with** $b \geq 0$**.**

Theorem **The above uncertain dynamical system is globally stabilizable by the feedback mechanism if and only if** $\boldsymbol{b < 3}$**.**

(Li & Guo, Automatica, 2010)

A General Polynomial Criterion

Consider additive nonlinear regression:

$$y_{t+1} = \theta^\tau f(y_t) + u_t + w_{t+1} \tag{1}$$

- $\theta \in \Theta \triangleq \{\theta \in \mathbb{R}^p : \|\theta\| \leq R\}$ **is a** p**-dimensional unknown parameter vector;**
- $\{w_t\}$ **is any bounded disturbance sequence, or a Gaussian white noise sequence;**
- $f(y_t) \triangleq [f_1(y_t), \cdots, f_p(y_t)]^\tau$ **belongs to:**

$$\mathcal{F}(b) = \left\{ f(\cdot) : f_i(x) = \Theta(|x|^{b_i}), \text{ as } x \to \infty \right\}$$

where $b = (b_1 \cdots b_p)$**, with** $b_1 > b_2 > \cdots > b_p > 0$**.**

With the exponents b_i introduced as above, define a characteristic polynomial:

$$P(z) = z^{p+1} - b_1 z^p + (b_1 - b_2) z^{p-1} + \cdots + (b_{p-1} - b_p) z + b_p$$

Theorem **Let $f \in \mathcal{F}(b)$ be a nonlinear function. Then the above uncertain nonlinear dynamical system with $\theta \in \Theta$ is globally stabilizable by the feedback mechanism if and only if**

$$P(z) > 0, \qquad \forall\, z \in (1, b_1)$$

(Xie and Guo, 1999; Li, Xie and Guo, 2006; Li and James, 2013)

Some Corollaries

Consider

$$y_{t+1} = \theta_1 {y_t}^{b_1} + \theta_2 {y_t}^{b_2} \cdots\cdots + \theta_p {y_t}^{b_p} + u_t + w_{t+1}$$

with $b_1 > b_2 > \cdots > b_p > 0$. **The polynomial criterion shows:**

- **For any** $\{b_i\}$ **with** $b_1 \geq 4$, **the system is not globally stabilizable;**
- **For any** $\{b_i\}$ **with** $\sum_{i=1}^{p} b_i < 4$, **the system is globally stabilizable;**
- **For any** $b_1 > 1$, **there always exit** p **and** $\{b_i\}$ **such that the corresponding system is not globally stabilizable.**

$\Longrightarrow$ **The class of uncertain nonparametric functions with growth rate faster than linear, is not globally stabilizable !**

A General Characterization

General nonlinearly parameterized model:

$$y_{t+1} = f(\theta, y_t) + u_t + w_{t+1} \tag{2}$$

- $\theta \in \{\theta \in \mathbb{R}^p : \|\theta\| \leq R\}$ **is unknown vector;**
- $\{w_t\}$ **is any bounded noise sequence;**
- $f(\cdot,\cdot)$ **belongs to:**

$$\mathcal{F}(b) = \left\{ f(\cdot,\cdot) : f_i'(\theta, x) = \Theta(|x|^{b_i}), \ \textbf{as } x \to \infty \right\}$$

where $f_i'(\cdot,\cdot) \triangleq \frac{\partial f(\theta,\cdot)}{\partial \theta_i}$ **is the sensitivity function of the** i**-th component of** $f(\cdot,\cdot)$**, and** $b = (b_1 \cdots b_p)$**.**

Consider again the characteristic polynomial:

$$P(z) = z^{p+1} - b_1 z^p + (b_1 - b_2) z^{p-1} + \cdots + (b_{p-1} - b_p) z + b_p$$

Introduce

$$\Omega = \{b = (b_1, \cdots, b_p) : \quad b_1 > b_2 > \cdots > b_p > 0\}$$

Define two sets:

$$\Omega_s \triangleq \{b \in \Omega : P(z) > 0, \ \forall\, z \in (1, b_1)\}$$

$$\Omega_u \triangleq \{b \in \Omega : b_1 \geq 4\}$$

It can be shown that

$$\Omega_s \subset \Omega_u^c$$

Possibility *vs.* Impossibility

- **The above class of uncertain nonlinear systems is globally stabilizable by the feedback mechanism provided that $b \in \Omega_s$.**
- **The above class of uncertain nonlinear systems is not globally stabilizable by the feedback mechanism as long as $b \in \Omega_u$.**

Now, let's take Ω_α as any set of parameter b such that

$$\Omega_s \subset \Omega_\alpha \subset \Omega_u^c.$$

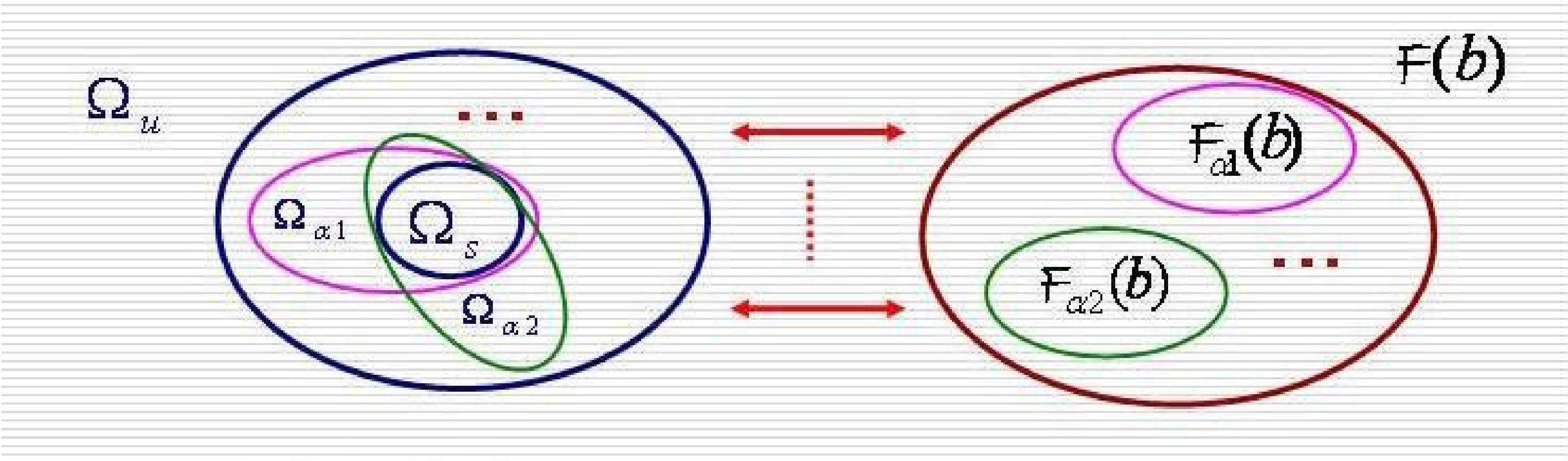
A General Characterization

Theorem **Consider the uncertain system with** $f(\cdot,\cdot) \in \mathcal{F}(b)$**. Then** $\mathcal{F}(b)$ **can be decomposed as**

$$\mathcal{F}(b) = \bigcup_{\alpha} F_{\alpha}(b),$$

where $\mathcal{F}_{\alpha}(b) \subset \mathcal{F}(b)$ **are disjoint and nonempty families of functions for different** α**, such that for each** $f(\cdot,\cdot) \in \mathcal{F}_{\alpha}(b)$**, the corresponding system is stabilizable by feedback if and only if** $b \in \Omega_{\alpha}$**.**

(Li & Guo, IEEE-TAC, 2011)



Ω_u
$\Omega_{\alpha 1}$
Ω_s
$\Omega_{\alpha 2}$
$\mathcal{F}(b)$
$\mathcal{F}_{\alpha 1}(b)$
$\mathcal{F}_{\alpha 2}(b)$

Rationale behind Impossibility

Let θ be a random vector independent of $\{w_t\}$. Consider

$$y_{t+1} = f(\theta, \phi_t) + w_{t+1},$$

where ϕ_t depends on $\{y_i, u_i, i \leq t\}$, and $\{w_t\}$ is white noise with variance σ_w^2. Let

$$\hat{f}(\theta, \phi_t) \triangleq E[f(\theta, \phi_t)|\mathcal{F}_t^y], \qquad \mathcal{F}_t^y \triangleq \sigma\{y_1, \cdots, y_t\}.$$

Then, we have

$$E[y_{t+1}^2|\mathcal{F}_t^y] = E[(f(\theta, \phi_t) - \hat{f}(\theta, \phi_t))^2|\mathcal{F}_t^y] + \hat{f}^2(\theta, \phi_t) + \sigma_w^2.$$

How to get a universal and valuable lower bound?

Conditional C-R inequality

Let θ and x be some random vectors and $p(x,\theta)$ the joint p.d.f.. Under some regularity conditions, for any measurable function $g(x,\theta)$,

$$\begin{aligned} & E_x\{[g(x,\theta) - E_x g(x,\theta)][g(x,\theta) - E_x g(x,\theta)]^T\} \\ \geq \quad & E_x \frac{\partial g(x,\theta)}{\partial \theta} \left\{ -E_x \left[\frac{\partial^2 \log p(x,\theta)}{\partial \theta^2} \right] \right\}^{-1} E_x^T \frac{\partial g(x,\theta)}{\partial \theta}, \end{aligned}$$

where $E_x y \triangleq E\{y|x\}$.

(cont.)

Conditional C-R inequality

Let θ and x be some random vectors and $p(x, \theta)$ the joint p.d.f. Under some regularity conditions, for any measurable function $g(x, \theta)$,

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where $E_x y \triangleq E\{y|x\}$.

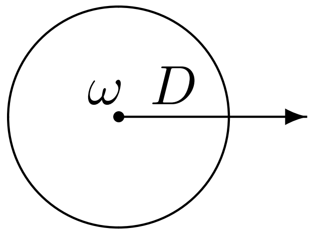
How to estimate the Fisher information matrix?

How to realize the above idea in a deterministic setting?

Stochastic Imbedding

Deterministic Framework $\longrightarrow$ **Imbed** $(\Omega, \mathcal{F}, P)$ $\longrightarrow$ **Stochastic Framework**

$P(D) > 0!$



$\theta(\omega), \{w_t(\omega)\}$

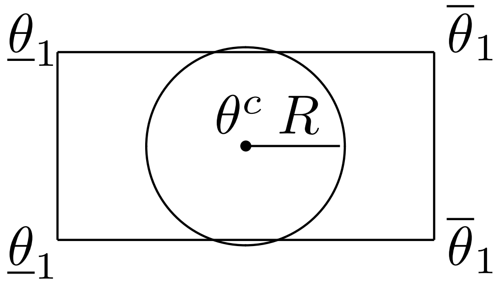
Deterministic Framework $\longleftarrow$ **Exist** $\theta, \{w_t\}$ $\longleftarrow$ **Stochastic Framework**

(Remark: The stochastic imbedding may not be effective for establishing possibility results, due to the well-known "exceptional set" problem).

Parameter Distribution Imbedding

Take θ to have the following spherical p.d.f.:

$$p(\theta) = \begin{cases} c(2^{-1}R^2 - \|\theta\|^2) & \textbf{if } 0 \leq \|\theta\| \leq R/2; \\ c(R - \|\theta\|)^2 & \textbf{if } R/2 \leq \|\theta\| \leq R; \end{cases}$$

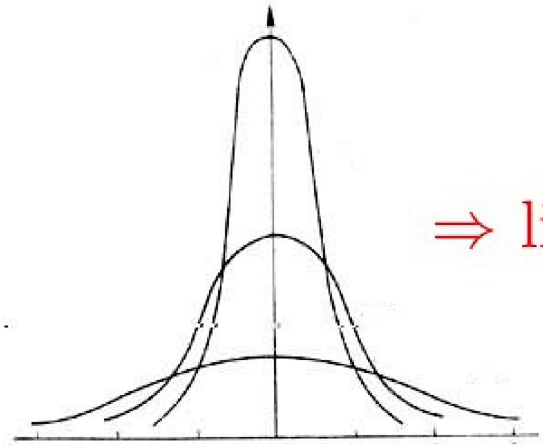


$\Rightarrow \|\theta\| \leq R$

Noise Distribution Imbedding

Take $\{w_t\}$ to be i.i.d and independent of θ with p.d.f.

$$q_t(x) = \frac{t}{\sqrt{2\pi}} \exp\left(-\frac{x^2 t^2}{2}\right)$$



$\Rightarrow \lim_{t \to \infty} w_t = 0$

Lower Bound to Conditional Variance

With the above stochastic imbedding, the Fisher information matrix can be calculated, leading to

$$E_x[f(\theta,\phi_t)-\hat{f}(\theta,\phi_t)]^2 \geq \frac{1}{2}E_x^{\tau}f'(\theta,\phi_t)P_t^{-1}(\theta)E_x f'(\theta,\phi_t),$$

where $x \triangleq \{y_1,\cdots,y_t\}$ **and**

$$P_{t+1}(\theta) \triangleq KI + M_1(t+1)^4\sum_{i=0}^{t} E[f'(\theta,\phi_i)f^{'\tau}(\theta,\phi_i)|\mathcal{F}_t^y].$$

which is a key step in establishing the impossibility, followed by a meticulous analysis of the involved nonlinear dynamical inequalities, and finally arriving at a connection with the polynomial criterion.

Case II. The Critical Value $\frac{3}{2} + \sqrt{2}$

The Critical Value $\frac{3}{2}+\sqrt{2}$

Consider the following nonparametric control system

$$y_{t+1} = f(y_t) + u_t + w_{t+1}, \qquad y_0 \in \mathbb{R}^1$$

with unknown function $f(\cdot) \in \mathcal{F} = \{$**all** $\mathbb{R}^1 \to \mathbb{R}^1$ **mappings**$\}$.
The Lipschitz norm on $\mathcal{F}$**:**

$$\|f\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

The set of uncertain functions:

$$\mathcal{F}(L) = \{f \in \mathcal{F} : \ \|f\| \leq L\}$$

L**: Serves as a measure of uncertainty**

Theorem. **The above class of uncertain dynamical systems described by $\mathcal{F}(L)$ is globally stabilizable by the feedback mechanism if and only if**

$$L < \frac{3}{2} + \sqrt{2}$$

- **If $L < \frac{3}{2} + \sqrt{2}$, then there is a feedback law $\{u_t\}$ such that the system is globally stable for any $f \in \mathcal{F}(L)$;**

- **If $L \geq \frac{3}{2} + \sqrt{2}$, then for any feedback law $\{u_t\}$, there is at least one system $f(\cdot) \in \mathcal{F}(L)$, such that the corresponding closed-loop system is unstable.**

(Xie & Guo, IEEE-TAC, 2000)

Why $\frac{3}{2} + \sqrt{2}$ is critical?

Let $\{y_t\}$ **be any sequence satisfying**

$$|y_{t+1} - (\textbf{center})_t| \leq L|y_t - (\textbf{neighbor})_{t-1}|,$$

where

$$(center)_t = \frac{1}{2}\left(\min_{0\leq i\leq t} y_i + \max_{0\leq i\leq t} y_i\right), \quad (neighbor)_{t-1} = \operatorname*{argmin}_{0\leq i\leq t-1} |y_t - y_i|.$$

Then, it follows that

$$\{y_t\} \textbf{ bounded} \iff L < \frac{3}{2} + \sqrt{2}$$

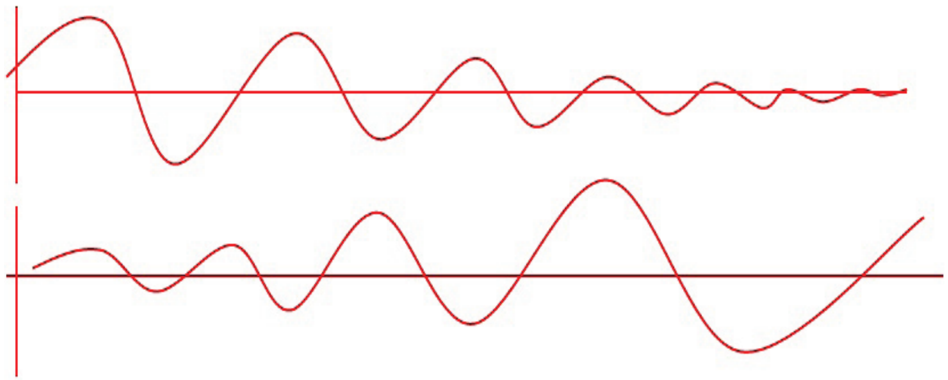
Another Related Fact

All solutions of

$$a_{n+1} = L(a_n - a_{n-1}) + \frac{1}{2}a_n$$

either converge to zero or oscillate about zero:

$$\Longleftrightarrow L < \frac{3}{2} + \sqrt{2}.$$



A General Theorem

Let $\{g(\theta, \cdot),\ \theta \in \Theta\}$ **be a model class with modeling error** $f(\cdot) \in \mathcal{F}(L)$ **plus a bounded disturbance:**

$$y_{t+1} = g(\theta, \phi_t) + f(y_t) + w_{t+1}, \quad t \geq 0,$$

where $\phi_t = [y_t, y_{t-1}, \cdots, y_{t-p+1}, u_t, u_{t-1}, \cdots, u_{t-q+1}]^\tau$**. Assume that the systems is "minimum phase" and that the SF of** $g(\cdot, \cdot)$ **is bounded by linear growth, etc. We have**

Theorem. **The above uncertain system with** $\{(\theta, f) \in (\Theta, \mathcal{F}(L)\}$ **is globally stabilizable by the feedback mechanism if and only if**

$$L < \frac{3}{2} + \sqrt{2}$$

(Huang and Guo, Automatica, 2012)

Modeling vs. Feedback

Let $G(\cdot)$ be a real uncertain system and $g(\theta, \cdot)$ be a model class, then

$$G(\cdot) = g(\theta, \cdot) + f(\cdot), \qquad f(\cdot) \triangleq G(\cdot) - g(\theta, \cdot)$$

- **Modeling and feedback are two main techniques in dealing with uncertainties, and the above theorem quantitatively shows how modeling and feedback are complementary in control systems design.**
- **In particular, the limitations of feedback may be compensated by improving the quality of modeling, by either understanding more about the concrete system mechanism, or choosing a more powerful parametric structure, or by both.**

Case III. Sampled-Data Feedback

Sampled-Data Feedback

Consider continuous-time nonlinear control systems:

$$\dot{x}_t = f(x_t) + u_t, \quad t \geq 0, \ x_0 \in R^1, \tag{3}$$

where the uncertain function f is locally Lipschitz.

The sampled-data feedback with sampling period h is defined simply by

$$u_t = u_{kh}, \quad t \in [kh, (k+1)h)$$

$$u_{kh} = g_k(x_0, x_h, ..., x_{kh}), \ \forall k \geq 0$$

where $g_k(\cdot)$ can be any Lebesgue measurable function.

Example 1. Stabilizing Feedback Under Sampling

Let θ be a scalar unknown parameter in

$$\dot{x}_t = \theta g(x_t) + u_t, \quad t \geq 0, \ x_0 \in R^1, \tag{4}$$

where g is locally Lipschitz and has the upper bound $|g(x)| \leq M|x|^b$, $b \geq 1$, $x \in R$. It is easy to show that the following continuous-time feedback is globally stabilizing:

$$u_t = -sgn(y_t)|y_t|^{b+\epsilon}, \quad \forall \epsilon > 0,$$

but the corresponding sampled feedback

$$u_t = -sgn(y_{kh})|y_{kh}|^{b+\epsilon}, \quad t \in [kh, (k+1)h), \ k = 0, 1, 2, ...$$

is not globally stabilizing, no matter how small the sampling period $h > 0$ is.

Example 2. Feedback Capability Under Sampling

Consider stochastic systems

$$\boldsymbol{dx_t = [f(x_t) + u_t]dt + \sigma dw_t, \quad t \geq 0,\ x_0 \in R^1}, \tag{5}$$

where $\boldsymbol{f}$ is locally Lipschitz, and $\{w_t\}$ is a standard Brownian motion. Assume that there are two constants $R > 0$ and $\delta > 0$ such that

$$xf(x) \geq |x|^{2+\delta}, \quad \delta > 0, \forall |x| \geq R.$$

Then, the stochastic system is not stabilizable by any sampled-data feedback, and in fact,

$$E|x_T|^2 = \infty, \quad \forall T > 0,\ x_0 \in R.$$

whatever how small the sampling period h is (even if f is known *a priori*).

An Impossibility Theorem

Consider the systems:

$$\dot{x}_t = f(x_t) + u_t, \quad t \geq 0, \ x_0 \in R^1, \tag{6}$$

where the local Lipschitz function f belongs to

$$G_L \triangleq \{f : \quad |f(x)| \leq L|x| + c, \quad \forall x \in R^1\}. \tag{7}$$

Theorem

If $Lh > 4.75$, then the uncertain system class G_L cannot be globally stabilized by any sampled-data feedback.

(Xue and Guo, 2002; Ren, Cheng and Guo, 2014)

IV. Concluding Remarks)

Concluding Remarks

- **All the impossibility theorems presented in this talk enjoy universality in the sense that they are actually valid for any larger class of uncertain systems and for any feedback laws.**
- **The main results indicate that the feedback capability depends on both information uncertainty and structural complexity, although our focus here is mainly placed on the former. The sensitivity function is more relevant than the range of unknown parameters.**

Concluding Remarks(Cont'd)

- **There are fundamental differences between continuous-time and discrete-time (or sampled-data)feedbacks for uncertain nonlinear systems in both results and analyses, where time lag in feedback is an inevitable feature for the later. Time-delay and time-varying systems may also be investigated.**

- **Finally, the investigation of the maximum feedback capability appears to be practically valuable, theoretically fundamental and mathematically challenging. This lecture only presents some preliminary results, and there is still a long way to go towards a complete understanding.**

Related Publications

- **L.Guo, "On critical stability of discrete-time adaptive nonlinear control", IEEE Trans. Automatic Control, Vol.42, No.11, pp.1488-1499, 1997.**
- **L.L.Xie and L.Guo, "Fundamental limitations of discrete-time adaptive nonlinear control", IEEE Trans. Automatic Control, Vol.44, No.9, pp.1777-1782, September, 1999.**
- **L.L.Xie and L.Guo, "How much uncertainty can be dealt with by feedback?", IEEE Trans. Automatic Control, Vol.45, No.12, 2000, pp.2203-2217.**
- **L.L.Xie and L.Guo, "Adaptive control of discrete-time nonlinear systems with structural uncertainties', AMS/IP studies in Advanced Mathematics, Vol.17,2000.**
- **F.Xue and L.Guo,"Necessary and sufficient conditions for adaptive stabilizability of jump linear systems", Communications in Information and Systems, Vol.1, No.2, pp.205-224, 2001.**
- **F.Xue and L.Guo, "On limitations of the sampled-data feedback for nonparametric dynamical systems", J. Systems Science and Complexity, Vol.15, No.3,pp.225-250, 2002.**

- **Y.X.Zhang and L.Guo, "A limit to the capability of feedback", IEEE Trans. Automatic Control,Vol.47, No.4, pp.687-692, 2002.**
- **C.Y.Li, L.L.Xie and L.Guo, "A polynomial criterion for adaptive stabilizability of discrete-time nonlinear systems", Communications in Information and Systems, Vol 6, No. 4, pp.273-298, 2006.**
- **C.Y.Li and L.Guo, "A new critical theorem for adaptive nonlinear stabilization", Automatica, Vol.46, pp.999-1007, June, 2010**
- **C.Y.Li and L.Guo, "On feedback capability in nonlinearly parameterized uncertain dynamical systems", IEEE Transactions Automatic Control, Vol.56, No.12, pp.2946-2951, 2011.**
- **C.D.Huang and L.Guo,"On feedback capability for a class of semiparametric uncertain systems", Automatica, Vol.48, No.5,873-878, 2012.**
- **J.L.Ren, Z.B.Cheng and L.Guo, "Further results on limitations of sampled-data feedback", J.Systems Science and Complexity, 2014**
- **M.Luo and L.Guo, "On the capability of feedback for unknown nonparametric systems with input delay", Proc.Chinese Control Conference, pp.2604-2607, 2014.**

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